MATH 2050C Lecture 10 (Feb 17)

[Problem set 5 posted, due on Feb 25.]

Last time: "Limit Thms" Assume $\lim(x_n)$, $\lim(y_n)$ exist.

$$\left\{\begin{array}{l} \lim(x_n \pm y_n) = \lim(x_n) \pm \lim(y_n) \\ \lim(x_n y_n) = \lim(x_n)\cdot\lim(y_n) \\ \lim\left(\dfrac{x_n}{y_n}\right) = \dfrac{\lim(x_n)}{\lim(y_n)} \quad (y_n \neq 0,\ \lim(y_n) \neq 0) \end{array}\right.$$

$$\left\{\begin{array}{l} \text{If } x_n \leq y_n \quad \forall n \in \mathbb{N} \\ \text{then } \lim(x_n) \leq \lim(y_n) \end{array}\right.$$

can be replaced by $\forall n \geq K$ for some K

Q: Are there other ways to prove $\lim(x_n)$ exist without using the ε-K definition?

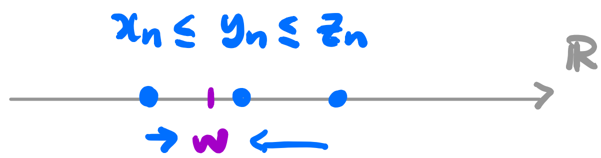
Thm: "Squeeze / Sandwich Theorem"

Let $(x_n), (y_n), (z_n)$ be seq. of real numbers s.t.

(1) $x_n \leq y_n \leq z_n \quad \forall n \in \mathbb{N}$ (or $\forall n \geq K$ for some K)

(2) $\lim(x_n) = w = \lim(z_n)$

THEN: $\lim(y_n) = w$



Remark: We do NOT need to assume $\lim(y_n)$ exists, it follows from the theorem.

E.g.) $\lim\left(\dfrac{\sin n}{n}\right) = 0$ since $\underset{\to 0}{-\dfrac{1}{n}} \leq \dfrac{\sin n}{n} \leq \underset{\to 0}{\dfrac{1}{n}}$

Proof: Let $\varepsilon > 0$ be fixed but arbitrary.

$\lim(x_n) = w \Rightarrow \exists K_1 \in \mathbb{N}$ s.t. $|x_n - w| < \varepsilon \quad \forall n \geq K_1$

$\lim(z_n) = w \Rightarrow \exists K_2 \in \mathbb{N}$ s.t. $|z_n - w| < \varepsilon \quad \forall n \geq K_2$

Take $K := \max\{K_1, K_2\}$, then $\forall n \geq K$, we have

$$-\varepsilon \underset{\because K \geq K_1}{<} x_n - w \leq y_n - w \leq z_n - w \underset{\because K \geq K_2}{<} \varepsilon$$

by (1)

i.e. $|y_n - w| < \varepsilon$. $\square$

Thm: "Ratio test"

Let (x_n) be a seq. of real numbers s.t.

(1) $x_n > 0 \quad \forall n \in \mathbb{N}$

(2) $\lim\left(\frac{x_{n+1}}{x_n}\right) = L < 1$

THEN: $\lim(x_n) = 0$

Model case:

geometric seq.

$(ar^n) \longrightarrow 0$

when $|r| < 1$

E.g.) Consider $(x_n) = \left(\frac{n}{2^n}\right)$, then

$$\left(\frac{x_{n+1}}{x_n}\right) = \left(\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}\right) = \left(\frac{n+1}{n} \cdot \frac{1}{2}\right) \longrightarrow \frac{1}{2} < 1$$

Apply Ratio test, $\lim(x_n) = 0$. $\square$

Proof: Idea: compare (x_n) with another geometric seq. (b^n) where $0<b<1$, apply sandwich thm.

Since $L<1$, $\exists\, r \in \mathbb{R}$ s.t.

$$L < r < 1$$

$$\lim\left(\frac{x_{n+1}}{x_n}\right) = L < 1$$

$$\Downarrow$$

$\frac{x_{n+1}}{x_n} \approx L$ for large n

$$x_{n+1} \approx L x_n$$

$$x_{n+2} \approx L x_{n+1} \approx L^2 x_n$$

$$x_{n+k} \approx L^k x_n$$

Because $\lim\left(\frac{x_{n+1}}{x_n}\right) = L$,

take $\varepsilon := r - L > 0$, by (2),

$\exists\, K \in \mathbb{N}$ s.t. $\forall\, n \geq K$,

$$\left|\frac{x_{n+1}}{x_n} - L\right| < \varepsilon = r - L$$

$$\Rightarrow \quad 0 < \frac{x_{n+1}}{x_n} < L + (r-L) = r < 1$$

$\because (1)$

In summary, we have

$$0 < x_{n+1} < r x_n, \quad \forall\, n \geq K$$

i.e. $0 < x_n < r x_{n-1} < r^2 x_{n-2} < \cdots < r^{n-K} x_K$

__Note__: $\lim\limits_{n\to\infty} (r^{n-K} x_K) = 0$ since it is a geom seq with $0<r<1$

By Sandwich Thm, $\lim(x_n) = 0$ $\blacksquare$

<u>Remark</u>: The condition $L < 1$ is crucial.

In general, the theorem does <u>NOT</u> hold when $L = 1$.

Consider the following example.

$$(x_n) := (n) \quad \text{divergent} \quad (\because \text{unbdd})$$

$$\text{but} \quad \left(\frac{x_{n+1}}{x_n}\right) = \left(\frac{n+1}{n}\right) \longrightarrow L = 1$$